DECOMPOSING THE C^*-ALGEBRAS OF GROUPOID EXTENSIONS

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ABSTRACT. We decompose the full and reduced C^*-algebras of an extension of a groupoid by the circle into a direct sum of twisted groupoid C^*-algebras.

1. INTRODUCTION

Let H be a compact group and denote by $\hat{H}$ the collection of equivalence classes of the irreducible unitary representations of H. The Peter-Weyl Theorem implies first, that every irreducible unitary representation of H is finite-dimensional, and, second, that the left-regular representation λ of H on $L^2(H)$ is unitarily equivalent to the direct sum $\oplus_{[U]\in\hat{H}} d_U \cdot U$, where d_U is the dimension of U. The C^*-algebra $C^*(H)$ of H is universal for the unitary representations of H, which means roughly that the unitary representations U of H are in one-to-one correspondence with the nondegenerate representations π_U of $C^*(H)$ on the Hilbert space of U. Since H is compact, the left-regular representation π_λ is an isomorphism, and the reduced C^*-algebra $C^*_r(H) := \pi_\lambda(C^*(H))$ coincides with $C^*(H)$. So in the C^*-setting, the Peter-Weyl theorem says that $C^*(H) = C^*_r(H)$ is a direct sum $\oplus_{[U]\in\hat{H}} M_{d(U)}(\mathbb{C})$ of matrix algebras.

A similar result holds for extensions of locally compact groups H by the circle $\mathbb{T}$. Let $\omega : H \times H \to \mathbb{T}$ be a continuous 2-cocycle. We associate two C^*-algebras to the pair (H, ω). The first is the twisted group C^*-algebra $C^*(H, \omega)$ which is universal for the ω-representations of H. For the second, equip $H^\omega := \mathbb{T} \times H$ with the product topology and multiplication $(s, \eta)(t, \gamma) = (st\omega(\eta, \gamma), \eta\gamma)$; then H^ω is a locally compact group and has a C^*-algebra. It follows from [20], for example, that $C^*(H^\omega)$ is isomorphic to the direct sum $\oplus_{n\in\mathbb{Z}} C^*(H, \omega^n)$ of twisted group C^*-algebras (see also [14, Corollary 3]). In this paper we generalize this latter result to locally compact Hausdorff groupoids.

Let G be a locally compact Hausdorff groupoid and $\omega : G^{(2)} \to \mathbb{T}$ a continuous 2-cocycle on the set $G^{(2)}$ of composable pairs in G. We show that the C^*-algebra of the extension G^ω is isomorphic to the direct sum $\oplus_{n\in\mathbb{Z}} C^*(G, \omega^n)$ of twisted groupoid C^*-algebras, and that this isomorphism factors through to the reduced C^*-algebras (see Theorems 3.2 and 4.1). The full twisted groupoid C^*-algebras have been used in [13], [10] and [4] to characterize when groupoid C^*-algebras have continuous trace or bounded trace, and their non-selfadjoint subalgebras have been studied in [11]. The reduced twisted groupoid C^*-algebras appear as the

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C^*-algebras with diagonal subalgebras in [9] and as the C^*-algebras with Cartan subalgebras in [18]. For example, if A is a C^*-algebra with diagonal subalgebra B then Kumjian's Theorem 3.1 of [9] implies that there exists a principal étale groupoid G and an extension of G by $\mathbb{T}$ implemented by a (possibly Borel) cocycle ω such that A is isomorphic to $C_r^*(G, \overline{\omega})$, and the isomorphism maps the diagonal B to a diagonal in $C_r^*(G, \overline{\omega})$. A similar result holds for Cartan subalgebras, except there the groupoid G may be only topologically principal [18, §5].

The main theorems of this paper provide a general framework for investigating twisted groupoid C^*-algebras using the literature on the non-twisted case. For example, suppose that G is principal. Then we deduce in Proposition 3.9 that $C^*(G)$ has continuous trace if and only if $C^*(G^\omega)$ has continuous trace, and if $C^*(G)$ has continuous trace then so does $C^*(G,\omega)$. See Proposition 3.10 for more results along these lines. We also deduce in Corollary 4.3 that if a groupoid G is amenable then $C^*(G,\omega)$ and $C_r^*(G,\omega)$ are isomorphic.

2. Preliminaries

Throughout, G is a second-countable, locally compact, Hausdorff groupoid with Haar sytstem $\{\lambda^u\}_{u\in G^{(0)}}$. We denote by λ_u the image of λ^u under inversion. We write $G^{(0)}$ for the unit space of G, $r = r_G, s = s_G : G \to G^{(0)}$ for the range and source maps $r_G(\gamma) = \gamma\gamma^{-1}$ and $s_G(\gamma) = \gamma^{-1}\gamma$, respectively, and $G^{(2)} := \{(\gamma, \eta) : s_G(\gamma) = r_G(\eta)\}$ for the set of composable pairs.

2.1. **Groupoid extensions.** Let $\omega : G^{(2)} \to \mathbb{T}$ be a continuous 2-cocycle, so that ω satisfies the cocycle identity

$$\omega(\gamma, \eta)\omega(\gamma\eta, \xi) = \omega(\eta, \xi)\omega(\gamma, \eta\xi).$$

We will assume throughout that ω is normalized in the sense that

$$\omega(r_G(\gamma), \gamma) = 1 = \omega(\gamma, s_G(\gamma)) \text{ and } \gamma \in G;$$

since every 2-cocycle is cohomologous to a normalized one[1] and because the associated C^*-algebras depend only on the class of the 2-cocycle (see [16, Proposition II.1.2]), there is no loss of generality. Following [16, page 73] we denote by G^ω the extension of G by $\mathbb{T}$ defined by ω: thus G^ω is the groupoid $\mathbb{T} \times G$ with the product topology, with range and source maps $r_{G^\omega}(t, \gamma) = (1, r_G(\gamma))$ and $s_{G^\omega}(t,\gamma) = (1, s_G(\gamma))$, multiplication $(s, \eta)(t, \gamma) = (st\omega(\eta, \gamma), \eta\gamma)$ and inverse $(t,\gamma)^{-1} = (t^{-1}\omega(\gamma,\gamma^{-1})^{-1}, \gamma^{-1})$. We identify the unit space of G^ω with $G^{(0)}$ via $(1, u) \mapsto u$. We say that G^ω is the *groupoid extension associated to* (G, ω). When we want to emphasize the product nature of G^ω we will denote it by $\mathbb{T} \times_\omega G$.

In order to reconcile our work with the literature, suppose that

$$G^{(0)} \to \mathbb{T} \times G^{(0)} \overset{i}{\hookrightarrow} E \overset{j}{\to} G := E/\mathbb{T} \to G^{(0)} \tag{2.1}$$

is an extention of topological groupoids such that i induces a free action of $\mathbb{T}$ on E by $t \cdot \gamma = i(t, r_E(\gamma))\gamma$ for $\gamma \in E$ and $t \in \mathbb{T}$. In [13, page 131], Muhly and Williams discuss a correspondence between extensions E and Borel 2-cocycles defined using a Borel cross section of j. They show that the 2-cocycle ω associated to an extension E is continuous if and only if there exists a continuous section of j, and then E is topologically isomorphic to G^ω. (If the cocycle ω is not continuous, then G^ω

[1]The coboundary implementing the equivalence is the image under the boundary map of the function $b(\gamma) = \omega(r_G(\gamma), \gamma)$.

is a Borel groupoid and the topological groupoid E is Borel isomorphic to G^ω.) When $E = G^\omega$ for a continuous ω, the inclusion i in (2.1) is $i(t,u) = (t,u)$ and j is the projection onto G. Furthermore, since ω is normalized, the action of $\mathbb{T}$ on G^ω induced by i is $s \cdot (t,\gamma) = (st,\gamma)$. Our main reason for restricting our attention to extensions associated to continuous cocycles is that we use the continuity in an apparently essential way in Lemma 3.6.

Example 2.1. It is quite easy to construct groupoids G with non-trivial continuous 2-cocycles ω, and hence there are many non-trivial extensions G^ω as described above. For example, take $X = S^3$ and recall that the Čech cohomology group $H^3(X,\mathbb{Z})$ is non-trivial. Since $H^3(X,\mathbb{Z})$ is isomorphic to the second sheaf cohomology group $H^2(X,\mathcal{S})$ (see, for example, [19, Theorem 4.42]), there exists a non-trivial cocycle $\lambda := \{\lambda_{ijk} : U_{ijk} \to \mathbb{T}\}$ where the U_{ijk} are the triple overlaps of an open cover $\{U_i\}$ of X. Define

$$\Psi : \bigcup_i U_i \times \{i\} \to X \text{ by } \Psi(x,i) = x \text{ for } x \in U_i.$$

Then $R(\Psi) := \{((x,i),(x,j)) : x \in U_i \cap U_j\}$ becomes a groupoid with range and source maps given by $r_{R(\Psi)}((x,i),(x,j)) = (x,i)$ and $s_{R(\Psi)}((x,i),(x,j)) = (x,j)$, multiplication defined by $((x,i),(x,j))((x,j),(x,k)) = ((x,i),(x,k))$ and inverse $((x,i),(x,j))^{-1} = ((x,j),(x,i))$. Now define $\omega_\lambda : R(\Psi)^{(2)} \to \mathbb{T}$ by

$$\omega_\lambda\big(((x,i),(x,j)),((x,j),(x,k))\big) = \lambda_{ijk}(x)$$

for $x \in U_{ijk}$. It is straightforward to check that ω_λ is a non-trivial continuous 2-cocyle (the $\lambda_{ijk} : U_{ijk} \to \mathbb{T}$ are continuous by definition, and ω_λ is a coboundary if and only if λ is).

Recall that a groupoid G is *principal* if the map $\Phi : \gamma \mapsto (r_G(\gamma), s_G(\gamma))$ is injective and is *proper* if Φ is proper. We say G is *transitive* if given $u,v \in G^{(0)}$ there exists $\gamma \in G$ such that $r_G(\gamma) = u$ and $s_G(\gamma) = v$.

Remark 2.2. Although Example 2.1 shows that there are many examples of non-trivial continuous 2-cocycles, principal transitive groupoids have only trivial ones. To see this, let G be a principal and transitive groupoid, and let ω be a normalized 2-cocycle on G. Pick $u \in G^{(0)}$, and let $b : \gamma \mapsto \omega(\gamma,\alpha_\gamma)$ where α_γ is the unique element such that $r_G(\alpha_\gamma) = s_G(\gamma)$ and $s_G(\alpha_\gamma) = u$. Then $\omega(\gamma,\eta) = b(\gamma)b(\eta)\overline{b(\gamma\eta)}$ and thus ω is a coboundary.

2.2. **The C^*-algebras.** Let E be a second-countable, locally compact groupoid with left Haar system β, and $\sigma : E^{(2)} \to \mathbb{T}$ a continuous, normalized 2-cocycle. For $f,g \in C_c(E)$, the formulas

$$f * g(\gamma) = \int_E f(\eta)g(\eta^{-1}\gamma)\sigma(\eta,\eta^{-1}\gamma)\, d\beta^{r_E(\gamma)}(\eta) \quad\text{and}\quad f^*(\gamma) = \overline{f(\gamma^{-1})}\,\overline{\sigma(\gamma,\gamma^{-1})}$$

define a convolution and involution on $C_c(E)$. These operations make $C_c(E)$ into a $*$-algebra, denoted by $C_c(E,\sigma)$. We denote by $\operatorname{Rep}(C_c(E,\sigma))$ the set of Hilbert-space representations $\rho : C_c(E,\sigma) \to B(\mathcal{H})$ that are continuous for the inductive limit topology on $C_c(E)$ and the weak operator topology on $B(\mathcal{H})$. Then

$$\|f\| = \sup\{\|\rho(f)\| : \rho \in \operatorname{Rep}(C_c(E,\sigma))\} \tag{2.2}$$

is finite and defines a pre-C^*-norm on $C_c(E,\sigma)$; the *twisted groupoid C^*-algebra* $C^*(E,\sigma)$ is defined as the completion of $C_c(E,\sigma)$ in this norm. (All of this is

non-trivial. If $\rho \in \operatorname{Rep}(C_c(E,\sigma))$ then ρ is the integrated form of a unitary representation of G by [17, Thèoréme 4.1(i)], and then ρ is bounded in Renault's I-norm [16, Proposition II.1.7]. Representations bounded by the I-norm are continuous in the inductive limit topology. It now follows that (2.2) defines a norm by [16, Proposition II.1.11] and that the definition of $C^*(E,\sigma)$ above coincides with the one in [16, Definition II.1.12].) If the cocycle is identically 1 then we write $C^*(E)$ for $C^*(E,1)$ and call it the *groupoid C^*-algebra* of E.

Let τ be normalized left Haar measure on $\mathbb{T}$; we will denote $d\tau(t)$ by dt. Let ω be a continuous 2-cocycle on G and let G^ω be the associated groupoid extension. Since $G^\omega = \mathbb{T}\times_\omega G$ has the product topology, the product measures $\{\tau\times\lambda^u : u \in G^{(0)}\}$ define a Haar system on the extension G^ω. For fixed $n\in\mathbb{Z}$, let

$$C_c(G^\omega,n) = \{f\in C_c(G^\omega) : f(s\cdot(t,\gamma)) = s^{-n}f(t,\gamma)\}.$$

As above, we denote by $\operatorname{Rep}(C_c(G^\omega,n))$ the set of Hilbert-space representations $\rho : C_c(G^\omega,n)\to B(\mathcal{H})$ that are continuous for the inductive limit topology on $C_c(G^\omega,n)$ and the weak operator topology on $B(\mathcal{H})$. Then $C_c(G^\omega,n)$ is a $*$-subalgebra of $C_c(G^\omega)$, and, as in [17, §5] and [13, page 130], the C^*-algebra $C^*(G^\omega,n)$ is the completion of $C_c(G^\omega,n)$ in the norm $\|f\| = \sup\{\|\rho(f)\| : \rho\in \operatorname{Rep} C_c(G^\omega,n)\}$. (Again, this is non-trivial: Corollaire 4.8 of [17] implies that this indeed defines a norm bounded by the I-norm.) The C^*-algebra $C^*(G^\omega,n)$ was studied in [17, §1], and, when $n=-1$, in [13].

Remark 2.3. Let $f,g\in C_c(G^\omega,n)$. Since the Haar system $\{\tau\times\lambda^u\}$ on G^ω is pulled back from the one on G and τ is normalized, the convolution $f*g$ can be written as an integral over G: a direct calculation shows that for any $s\in\mathbb{T}$,

$$\begin{aligned}
f*g(t,\gamma) &:= \int_G\int_{\mathbb{T}} f(r,\eta)g((r,\eta)^{-1}(t,\gamma))\,dr\,d\lambda^{r_G(\gamma)}(\eta)\\
&= \int_G\int_{\mathbb{T}} r^{-n}r^n f(1,\gamma)g(t\omega(\eta,\eta^{-1})^{-1}\omega(\eta^{-1},\gamma),\eta^{-1}\gamma)\,dr\,d\lambda^{r_G(\gamma)}(\eta)\\
&= \int_G f(s,\eta)g((s,\eta)^{-1}(t,\gamma))\;d\lambda^{r_G(\gamma)}(\eta).
\end{aligned}$$

3. Decomposing the C^*-algebra of a groupoid extension

Throughout $\omega : G^{(2)}\to\mathbb{T}$ is a continuous normalized 2-cocycle, and G^ω is the groupoid extension associated to (G,ω). Note that ω^n is also a continuous 2-cocycle. The goal of this section is to prove that $C^*(G^\omega)$ is isomorphic to a direct sum of twisted groupoid C^*-algebras $C^*(G,\omega^n)$. We start by proving that $C^*(G^\omega,n)$ is a quotient of $C^*(G^\omega)$ and is isomorphic to $C^*(G,\omega^n)$.

Lemma 3.1. *Let G^ω be the groupoid extension associated to (G,ω). Fix $n\in\mathbb{Z}$.*

(a) [17, Lemma 3.3] *Define $\chi_n : C_c(G^\omega)\to C_c(G^\omega,n)$ by*

$$\chi_n(f)(t,\gamma) := \int_{\mathbb{T}} f(s\cdot(t,\gamma))s^n\,ds = \int_{\mathbb{T}} f(st,\gamma)s^n\,ds.$$

Then χ_n is a $*$-homomorphism continuous with respect to the inductive limit topologies, and extends to a $*$-homomorphism $\chi_n : C^*(G^\omega)\to C^*(G^\omega,n)$ such that $\chi_n(f) = f$ for $f\in C_c(G^\omega,n)$. In particular, χ_n is a quotient map.

(b) Let $\phi_n : C_c(G^\omega,n)\to C_c(G,\omega^n)$ be the map $\phi_n(f)(\gamma) = f(1,\gamma)$ for $\gamma\in G$. Then ϕ_n extends to a $*$-isomorphism of $C^*(G^\omega,n)$ onto $C^*(G,\omega^n)$.

Proof. Part (a) is [17, Lemma 3.3] (see also [16, Proposition II.1.22] for a detailed proof of the case $n = 1$).

(b) It suffices to show that $\phi_n : C_c(G^\omega, n) \to C_c(G, \omega^n)$ is a continuous bijective $*$-homomorphism with a continuous inverse. For then ϕ_n and ϕ_n^{-1} extend to $*$-homomorphisms $\phi_n : C^*(G^\omega, n) \to C^*(G, \omega^n)$ and $\phi_n^{-1} : C^*(G, \omega^n) \to C^*(G^\omega, n)$, and by continuity $\phi_n \circ \phi_n^{-1} = \mathrm{id}$ and $\phi_n^{-1} \circ \phi_n = \mathrm{id}$, giving that ϕ_n is an isomorphism.

To see that ϕ_n is a homomorphism, we first need a calculation with cocycles. Let $\eta, \gamma \in G$ with $r_G(\gamma) = r_G(\eta)$. Since ω is normalized, we have

$$1 = \omega(r_G(\eta^{-1}\gamma), \eta^{-1}\gamma) = \omega(s_G(\eta), \eta^{-1}\gamma) = \omega(\eta^{-1}\eta, \eta^{-1}\gamma) \quad \text{and}$$
$$\omega(\eta, \eta^{-1}) = \omega(\eta, \eta^{-1})\omega(\eta\eta^{-1}, \eta) = \omega(\eta^{-1}, \eta)\omega(\eta, \eta^{-1}\eta) = \omega(\eta^{-1}, \eta).$$

Thus

$$\omega(\eta^{-1}, \eta) = \omega(\eta^{-1}, \eta)\omega(\eta^{-1}\eta, \eta^{-1}\gamma) = \omega(\eta, \eta^{-1}\gamma)\omega(\eta^{-1}, \gamma),$$

and it follows that

$$\overline{\omega(\eta, \eta^{-1})}\omega(\eta^{-1}, \gamma) = \overline{\omega(\eta, \eta^{-1}\gamma)}. \tag{3.1}$$

So, for $f, g \in C_c(G^\omega, n)$,

$$\begin{aligned}
\phi_n(f * g)(\gamma) = f * g(1, \gamma) &= \int_G f(1, \eta) g((1, \eta)^{-1}(1, \gamma))\, d\lambda^{r_G(\gamma)}(\eta) \\
&= \int_G f(1, \eta) g(\overline{\omega(\eta, \eta^{-1})}\omega(\eta^{-1}, \gamma), \eta^{-1}\gamma)\, d\lambda^{r_G(\gamma)}(\eta) \\
&= \int_G f(1, \eta) g(\overline{\omega(\eta, \eta^{-1}\gamma)}, \eta^{-1}\gamma)\, d\lambda^{r_G(\gamma)}(\eta) \quad \text{(using (3.1))} \\
&= \int_G f(1, \eta) g(1, \eta^{-1}\gamma)\omega(\eta, \eta^{-1}\gamma)^n\, d\lambda^{r_G(\gamma)}(\eta) \\
&= \phi_n(f) * \phi_n(g)
\end{aligned}$$

and

$$\begin{aligned}
\phi_n(f^*)(\gamma) = \overline{f((1, \gamma)^{-1})} &= \overline{f(\omega(\gamma, \gamma^{-1})^{-1}, \gamma^{-1})} \\
&= \overline{f(1, \gamma^{-1})\omega(\gamma, \gamma^{-1})^n} = \phi_n(f)^*(\gamma).
\end{aligned}$$

So ϕ_n is a $*$-homomorphism. To see ϕ_n is injective on $C_c(G^\omega, n)$, suppose $f(1, \gamma) = g(1, \gamma)$ for all $\gamma \in G$. Then for all $t \in \mathbb{T}$,

$$f(t, \gamma) = t^{-n} f(1, \gamma) = t^{-n} g(1, \gamma) = g(t, \gamma)$$

and thus $f = g$. To see ϕ_n is onto $C_c(G, \omega^n)$, let $f \in C_c(G)$ and note that $(t, \gamma) \mapsto t^{-n} f(\gamma)$ is in $C_c(G^\omega, n)$, and ϕ_n sends it back to f. So $\phi_n : C_c(G^\omega, n) \to C_c(G, \omega^n)$ is a bijection.

If $F_i \to F$ in the inductive limit topology on $C_c(G^\omega)$, then $F_i(1, \cdot) \to F(1, \cdot)$ uniformly on a fixed compact set as well. Thus ϕ_n is continuous for the inductive limit topology on $C_c(G^\omega, n)$ and extends to a $*$-homomorphism of the C^*-algebras. Similarly, if $f_i \to f \in C_c(G)$ in the inductive limit topology, then $|t^{-n} f_i(\gamma) - t^{-n} f(\gamma)| \leq |f_i(\gamma) - f(\gamma)|$ is eventually small, so that $\phi_n^{-1}(f_i) \to \phi_n^{-1}(f)$ in the inductive limit topology as well. As outlined at the beginning of the proof, this implies that ϕ_n extends to an isomorphism of $C^*(G^\omega, n)$ onto $C^*(G, \omega^n)$. $\square$

Define $\Upsilon_n := \phi_n \circ \chi_n : C^*(G^\omega) \to C^*(G, \omega^n)$; then

$$\Upsilon_n(F)(\gamma) = \int_{\mathbb{T}} F(t,\gamma) t^n \, dt \quad \text{for } F \in C_c(G^\omega).$$

Theorem 3.2. *Let G be a second-countable, locally compact Hausdorff groupoid with a Haar system λ. Let $\omega : G^{(2)} \to \mathbb{T}$ be a continuous 2-cocycle and let G^ω be the groupoid extension associated to (G, ω). Then the map $\Upsilon : C_c(G^\omega) \to \oplus_{n\in\mathbb{Z}} C_c(G, \omega^n)$ defined by $F \mapsto (\Upsilon_n(F))$ extends to an isomorphism of $C^*(G^\omega)$ onto $\oplus_{n\in\mathbb{Z}} C^*(G, \omega^n)$.*

To prove Theorem 3.2 we first prove that the subalgebra $I_n := \overline{C_c(G^\omega, n)}^{\|\cdot\|_{C^*(G^\omega)}}$ is an ideal of $C^*(G^\omega)$ which is isomorphic to $C^*(G^\omega, n)$, and second, that $C^*(G^\omega)$ is the (internal) direct sum of the I_n.

Lemma 3.3. *Let G^ω be the groupoid extension associated to (G, ω).*

(a) For $f \in C_c(G^\omega, n)$, $\|f\|_{C^*(G^\omega)} = \|f\|_{C^*(G^\omega, n)}$.
(b) The map $\chi_n : C_c(G^\omega, n) \subset C_c(G^\omega) \to C_c(G^\omega, n)$ extends to an isometry χ_n of the subalgebra I_n of $C^*(G^\omega)$ onto $C^*(G^\omega, n)$.
(c) The quotient map $\chi_n : C^*(G^\omega) \to C^*(G^\omega, n)$ is identically zero on I_m if $n \neq m$.

Proof. (a) Fix $f \in C_c(G^\omega, n)$. If $\pi \in \operatorname{Rep}(C_c(G^\omega, n))$ then by Lemma 3.1(a), $\pi \circ \chi_n \in \operatorname{Rep}(C_c(G^\omega))$. Since $f = \chi_n(f)$ we have

$$\begin{aligned} \|f\|_{C_c(G^\omega,n)} &= \sup\{\|\pi(f)\| : \pi \in \operatorname{Rep}(C_c(G^\omega, n))\} \\ &= \sup\{\|\pi \circ \chi_n(f)\| : \pi \in \operatorname{Rep}(C_c(G^\omega, n))\} \\ &\leq \|f\|_{C^*(G^\omega)}. \end{aligned}$$

Conversely, if $\rho \in \operatorname{Rep}(C_c(G^\omega))$ then $\rho|_{C_c(G^\omega,n)}$ is also continuous in the inductive limit topology on $C_c(G^\omega, n)$. Fix $\epsilon > 0$. Pick a representation ρ of $C_c(G^\omega)$ such that $\|f\|_{C^*(G^\omega)} < \|\rho(f)\| + \epsilon$. Then

$$\|f\|_{C^*(G^\omega)} < \|\rho(f)\| + \epsilon = \|\rho|_{C_c(G^\omega,n)}(f)\| + \epsilon \leq \|f\|_{C^*(G^\omega,n)} + \epsilon.$$

Thus $\|f\|_{C^*(G^\omega)} \leq \|f\|_{C^*(G^\omega,n)}$, and $\|f\|_{C^*(G^\omega)} = \|f\|_{C^*(G^\omega,n)}$ as desired.

(b) Fix $g \in I_n$. Let $\{f_i\} \subset C_c(G^\omega, n)$ be a sequence converging to g. By (a), $\|f_i\|_{C^*(G^\omega)} = \|\chi_n(f_i)\|_{C^*(G^\omega,n)}$, and hence $\|g\|_{C^*(G^\omega)} = \|\chi_n(g)\|_{C^*(G^\omega,n)}$. So χ_n is isometric on the subalgebra I_n of $C^*(G^\omega)$. Furthermore, $\chi_n|_{I_n}$ is onto since $C_c(G^\omega, n)$ is dense in $C^*(G^\omega, n)$. So $\chi_n|_{I_n}$ is an isomorphism.

(c) This is a direct calculation. □

Lemma 3.4. *Let G^ω be the groupoid extension associated to (G, ω). For each $n \in \mathbb{Z}$, I_n is an ideal in $C^*(G^\omega)$. Furthermore, $I_n I_m = \{0\}$ if $n \neq m$.*

Proof. Let $f \in C_c(G^\omega)$ and $g \in C_c(G^\omega, n)$. Then

$$\begin{aligned} f * g(s \cdot (t,\gamma)) &= \int_G \int_{\mathbb{T}} f(r,\eta) g((r,\eta)^{-1}(st,\gamma)) \, dr \, d\lambda^{r_G(\gamma)}(\eta) \\ &= \int_G \int_{\mathbb{T}} f(r,\eta) g(sr^{-1}t\overline{\omega(\eta,\eta^{-1})}\omega(\eta^{-1},\gamma), \eta^{-1}\gamma) \, dr \, d\lambda^{r_G(\gamma)}(\eta) \\ &= \int_G \int_{\mathbb{T}} f(r,\eta) g(s \cdot ((r,\eta)^{-1}(t,\gamma))) \, dr \, d\lambda^{r_G(\gamma)}(\eta) \end{aligned}$$

$$= s^{-n}\int_G\int_{\mathbb{T}} f(r,\eta)g((r,\eta)^{-1}(t,\gamma))\,dr\,d\lambda^{r_G(\gamma)}(\eta)$$
$$= s^{-n} f * g(t,\gamma).$$

Thus $f*g \in C_c(G^\omega, n) \subset I_n$. Since $C_c(G^\omega, n)$ is closed under involution $g*f \in I_n$ as well. Since I_n is closed the above calculations show that I_n is an ideal in $C^*(G^\omega)$.

To see that $I_mI_n = \{0\}$ unless $n = m$, let $f \in C_c(G^\omega, m), g \in C_c(G^\omega, n)$. Then

$$\begin{aligned}
f*g(t,\gamma) &= \int_G\int_{\mathbb{T}} f(r,\eta)g(r^{-1}t\overline{\omega(\eta,\eta^{-1})}\omega(\eta^{-1},\gamma),\eta^{-1}\gamma)\,dr\,d\lambda^{r_G(\gamma)}(\eta)\\
&= \int_G\int_{\mathbb{T}} r^{-m}f(1,\eta)r^n g(t\overline{\omega(\eta,\eta^{-1})}\omega(\eta^{-1},\gamma),\eta^{-1}\gamma)\,dr\,d\lambda^{r_G(\gamma)}(\eta)\\
&= \int_G f(1,\eta)g(t\overline{\omega(\eta,\eta^{-1})}\omega(\eta^{-1},\gamma),\eta^{-1}\gamma)\,d\lambda^{r_G(\gamma)}(\eta)\int_{\mathbb{T}} r^{n-m}\,dr.\\
&= \begin{cases}\int_G f(1,\eta)g(t\overline{\omega(\eta,\eta^{-1})}\omega(\eta^{-1},\gamma),\eta^{-1}\gamma)\,d\lambda^{r_G(\gamma)}(\eta) & \text{if } m=n\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$ □

Notation 3.5. For $f \in C_c(G)$, $\psi \in C(\mathbb{T})$, denote by $\psi\otimes f$ the function $(t,\gamma)\mapsto \psi(t)f(\gamma)$. In particular, for fixed m, we write $s^m\otimes f$ for the function $(t,\gamma)\mapsto t^m f(\gamma)$ in I_{-m}.

Lemma 3.6. *Let G^ω be the groupoid extension associated to (G,ω). The* $\operatorname{span}\{s^m\otimes f : m\in\mathbb{Z}, f\in C_c(G)\}$ *is dense in $C_c(G^\omega)$ in the inductive limit topology.*

Proof. Fix $F \in C_c(G^\omega)$ and $\epsilon > 0$. Let U_1 and U_2 be open, relatively compact neighborhoods in $\mathbb{T}$ and G, respectively, such that $\operatorname{supp} F \subset U_1 \times U_2$. Because ω is continuous, $G^\omega = \mathbb{T}\times_\omega G$ has the product topology, and the map $t\mapsto F(t,\cdot)$ is in $C_c(\mathbb{T}, C_c(G))$. So the support of $t\mapsto F(t,\cdot)$ is contained in U_1. For each $t\in U_1$ let

$$W_t := \{s\in\mathbb{T} : \|F(s,\cdot)-F(t,\cdot)\|_\infty < \epsilon/2\}\cap U_1.$$

Then W_t is an open cover of the compact set $\operatorname{supp}(t\mapsto F(t,\cdot))$, so there exists a finite subcover $W_{t_1},\dots W_{t_N}$. Let $\{\psi_i\}_{i=1}^N$ be a partition of unity subordinate to this cover. Since $\sum\psi_i(t)\le 1$ for all $t\in\mathbb{T}$,

$$\begin{aligned}
\|\sum_{i=1}^N \psi_i(t)F(t_i,\cdot)-F(t,\cdot)\|_\infty &= \|\sum_{i=1}^N\psi_i(t)F(t_i,\cdot)-\sum_{i=1}^N\psi_i(t)F(t,\cdot)\|_\infty\\
&\le \sum_{i=1}^N \psi_i(t)\|F(t_i,\cdot)-F(t,\cdot)\|_\infty < \frac{\epsilon}{2}.
\end{aligned}$$

For $\gamma\in\bigcup_{i=1}^N\operatorname{supp}(F(t_i,\cdot))\subset U_2$, let U_γ be the open set

$$U_\gamma := \{\eta\in G : |F(t_i,\gamma)-F(t_i,\eta)| < \epsilon/2 \text{ for } 1\le i\le N\}\cap U_2.$$

Since $\bigcup_{i=1}^N\operatorname{supp}(F(t_i,\cdot))$ is compact there exists a finite subcover $U_{\gamma_1},\dots,U_{\gamma_M}$. Let $\{f_j\}_{j=1}^M$ be a partition of unity subordinate to this subcover. For $\gamma\in G$ and each $i\in\{1,\dots,N\}$ we have

$$\Big|\sum_{j=1}^M f_j(\gamma)F(t_i,\gamma_j)-F(t_i,\gamma)\Big| = \Big|\sum_{j=1}^M f_j(\gamma)F(t_i,\gamma_j)-\sum_{j=1}^M f_j(\gamma)F(t_i,\gamma)\Big|$$

$$\leq \sum_{j=1}^{M} f_j(\gamma)\big|F(t_i,\gamma_j)-F(t_i,\gamma)\big| < \frac{\epsilon}{2}.$$

Now set $F_\epsilon := \sum_{i,j=1}^{M,N} F(t_i,\gamma_j)\psi_i\otimes f_j$, and note that $\operatorname{supp} F_\epsilon$ is contained in $U_1\times U_2$ by construction. We have

$$\|F_\epsilon - F\|_\infty = \sup_{(t,\gamma)}\{\big|\sum_{i,j} F(t_i,\gamma_j)\psi_i(t)f_j(\gamma) - F(t,\gamma)\big|\}$$

$$\leq \sup_{(t,\gamma)}\{\big|\sum_{i,j}\psi_i(t)f_j(\gamma)F(t_i,\gamma_j) - \sum_i \psi_i(t)F(t_i,\gamma) + \sum_i \psi_i(t)F(t_i,\gamma) - F(t,\gamma)\big|\}$$

$$< \sup_{(t,\gamma)}\{\sum_i \psi_i(t)\big|\sum_j f_j(\gamma)F(t_i,\gamma_j) - F(t_i,\gamma)\big| + \frac{\epsilon}{2}\} < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon.$$

We have now shown that $\operatorname{span}\{\psi\otimes f : \psi\in C(\mathbb{T}), f\in C_c(G)\}$ is dense in $C_c(G^\omega)$ in the inductive limit topology. Thus it follows from the Stone-Weierstrass theorem that $\operatorname{span}\{s^m\otimes f : m\in\mathbb{Z}, f\in C_c(G)\}$ is dense in $C_c(G^\omega)$. □

Lemmas 3.4 and 3.6 give:

Proposition 3.7. *Let G^ω be the groupoid extension associated to (G,ω). Then $C^*(G^\omega) = \oplus_{n\in\mathbb{Z}} I_n$.*

Proof of Theorem 3.2. Both $\chi_n : I_n \to C^*(G^\omega, n)$ and $\phi_n : C^*(G^\omega, n) \to C^*(G,\omega^n)$ are isomorphisms by Lemmas 3.3 and 3.1(b), so $\Upsilon_n|_{I_n} = \phi_n\circ\chi_n|_{I_n}$ is an isomorphism of I_n onto $C^*(G,\omega^n)$. But by Lemma 3.3(c), $\chi_n(I_m) = \{0\}$ if $n\neq m$, so Theorem 3.2 follows from Proposition 3.7. □

We now show that Theorem 3.2 leads to a general framework for deducing results about twisted groupoid C^*-algebras from untwisted ones. The basic idea is that many properties of principal groupoids are shared with their extensions by $\mathbb{T}$. We start with a general lemma. The *stabilizer subgroupoid* of a groupoid G is $\{\gamma\in G : r(\gamma) = s(\gamma)\}$ and $A_u := \{\gamma\in G : r(\gamma) = u = s(\gamma)\}$ is the *stability subgroup* at u.

Lemma 3.8. *Let G be a groupoid and G^ω be the extension associated to (G,ω). Let A and A^ω be the respective stabilizer subgroupoids of G and G^ω. Then the map $(t,\gamma)\mapsto[\gamma]$ induces a homeomorphism and isomorphism of G^ω/A^ω onto G/A.*

Proof. The stability subgroups of G^ω are $\mathbb{T}\times_\omega A_u$ where A_u is the stability subgroup of G at u. Thus $A^\omega = \cup_{u\in G^{(0)}}\mathbb{T}\times_\omega A_u = \mathbb{T}\times_\omega A$.

We will show that the map $f : (t,\gamma)\mapsto[\gamma]$ induces a homeomorphism and isomorphism of G^ω/A^ω onto G/A. Certainly f is a groupoid morphism, and is continuous and surjective. If $f(t,\gamma) = f(s,\delta)$ then there exists $\alpha\in A_{s(\gamma)}$ such that $\gamma = \delta\alpha$. Then $(t,\gamma) = (s,\delta)(s^{-1}t\overline{\omega(\delta,\alpha)},\alpha)$, and $(s^{-1}t\overline{\omega(\delta,\alpha)},\alpha)\in A^\omega$. Hence $[(t,\gamma)] = [(s,\delta)]$. So f induces a continuous bijection $\tilde f : G^\omega/A^\omega\to G/A$. Similarly, the function $g : G\to G^\omega/A^\omega$ defined by $g(\gamma) = [(1,\gamma)]$ induces a continuous bijection $\tilde g : G/A\to G^\omega/A^\omega$, and it is easy to check that $\tilde g$ is the inverse of $\tilde f$. Thus $\tilde f$ is a homeomorphism. □

Proposition 3.9. *Let G be a principal groupoid and let G^ω be the extension associated to a continuous 2-cocycle $\omega : G^{(2)}\to\mathbb{T}$. Then*

(a) $C^*(G)$ has continuous trace if and only if $C^*(G^\omega)$ has continuous trace; and
(b) if $C^*(G)$ has continuous trace then so does $C^*(G,\omega)$.

Proof. (a) First suppose that $C^*(G)$ has continuous trace. Since G is principal, [12, Theorem 2.3] implies that G is a proper groupoid. Now consider G^ω: since G is principal the stability subgroups of G^ω are $\mathbb{T} \times \{u\}$ where $u \in G^{(0)}$, and the stabilizer subgroupoid is $A^\omega = \mathbb{T} \times G^{(0)}$. In particular, the stability subgroups of G^ω are all abelian and $u \mapsto \mathbb{T} \times \{u\}$ is continuous in the Fell topology on the set of closed subgroups of G. By Lemma 3.8, the quotient groupoid G^ω/A^ω and G are homeomorphic, and hence G^ω/A^ω is proper. Now $C^*(G^\omega)$ has continuous trace by [10, Theorem 1.1].

Conversely, suppose $C^*(G^\omega)$ has continuous trace. By Theorem 3.2, $C^*(G) = C^*(G, \omega^0)$ is a direct summand of $C^*(G^\omega)$. Hence $C^*(G)$ has continuous trace by [15, Proposition 6.2.10].

(b) Suppose that $C^*(G)$ has continuous trace. Then $C^*(G^\omega)$ has continuous trace by (a). By Theorem 3.2, $C^*(G, \omega)$ is a direct summand of $C^*(G^\omega)$, and hence $C^*(G, \omega)$ has continuous trace as well. □

Many properties are shared by G and G^ω: having a Haar system, being Cartan, proper or integrable, and any topological property of the orbit spaces. This gives the proposition below. Example 3.11 below shows that we cannot expect to extend Propositions 3.9 and 3.10 to non-principal groupoids G.

Proposition 3.10. *Let G be a principal groupoid and let G^ω be the extension associated to a continuous 2-cocycle $\omega : G^{(2)} \to \mathbb{T}$.*

- *(a) $C^*(G)$ is a Fell algebra if and only if $C^*(G^\omega)$ is a Fell algebra. If $C^*(G)$ is a Fell algebra then so is $C^*(G, \omega)$.*
- *(b) $C^*(G)$ has bounded trace if and only if $C^*(G^\omega)$ has bounded trace. If $C^*(G)$ has bounded trace then so does $C^*(G, \omega)$.*
- *(c) $C^*(G)$ is liminal if and only if $C^*(G^\omega)$ is liminal. If $C^*(G)$ is liminal then so is $C^*(G, \omega)$;*
- *(d) $C^*(G)$ is postliminal if and only if $C^*(G^\omega)$ is postliminal. If $C^*(G)$ is postliminal then so is $C^*(G, \omega)$.*

Proof. Since G is principal the stability subgroups of G^ω are $\mathbb{T} \times \{u\}$ where $u \in G^{(0)}$; in particular they are abelian and vary continuously.

(a) and (b) We can proceed as in the proof of Proposition 3.9 replacing [10, Theorem 1.1] with [4, Theorem 6.5] and [4, Theorem 6.4], respectively.

(c) First suppose that $C^*(G)$ is liminal. Since the stability subgroups of G are trivial, the orbit space $G^{(0)}/G$ is T_1 by [3, Theorem 6.1]. But the orbit space of G^ω is homeomorphic to the orbit space of G via $[(1, u)] \mapsto [u]$, hence is T_1 as well. Since the stability subgroups of G^ω are amenable and liminal, $C^*(G^\omega)$ is liminal by [3, Theorem 6.1].

Second, suppose that $C^*(G^\omega)$ is liminal. By Theorem 3.2, $C^*(G) = C^*(G, \omega^0)$ is a direct summand of $C^*(G^\omega)$. Hence $C^*(G)$ is liminal by [15, Proposition 6.2.9]. This gives the first statement of (c).

Finally, suppose that $C^*(G)$ is liminal. Then $C^*(G^\omega)$ is liminal. By Theorem 3.2, $C^*(G, \omega)$ is a direct summand of $C^*(G^\omega)$, and hence $C^*(G, \omega)$ must be liminal as well. This gives the second statement of (c).

(d) Theorem 7.1 of [3] says that the groupoid C^*-algebra of a groupoid with amenable stability subgroups is postliminal if and only if the orbit space is T_0 and the stability subgroups are postliminal. So (d) follows as above using [3, Theorem 7.1] in place of [3, Theorem 6.1]. □

Example 3.11. When the groupoid G is not principal, the stability subgroups of G^ω can easily fail to be abelian, liminal or postliminal even if the stability subgroups of G are abelian, liminal or postliminal, respectively. Thus the theorems used to prove Propositions 3.9 and 3.10, such as [10, Theorem 1.1] and [3, Theorem 6.1], do not apply. The following is an example of a group G and a 2-cocycle ω such that $C^*(G)$ has continuous trace but $C^*(G,\omega)$ and $C^*(G^\omega)$ are not even postliminal. Thus we cannot expect an analog of Proposition 3.9 when the groupoid G is not principal.

Let $\theta \in (0,1)$ be irrational and define $\omega : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{T}$ by $\omega((m_1,m_2),(n_1,n_2)) = e^{-2\pi i m_1 n_2 \theta}$. The twisted group C^*-algebra $C^*(\mathbb{Z}^2,\omega)$ is isomorphic to the irrational rotation algebra $A_\theta = C(\mathbb{T}) \rtimes \mathbb{Z}$ (see, for example, [5, pp. 21-22]). Since θ is irrational the orbit space $\mathbb{T}/\mathbb{Z}$ is not T_0, and hence A_θ is not postliminal by [8, Theorem 3.3]. Thus $C^*(\mathbb{Z}^2,\omega)$ is not postliminal. By Theorem 3.2, $C^*(\mathbb{Z}^2,\omega)$ is a summand of $C^*((\mathbb{Z}^2)^\omega)$, so $C^*((\mathbb{Z}^2)^\omega)$ is not postliminal either. Thus $C^*(\mathbb{Z}^2,\omega)$ and $C^*((\mathbb{Z}^2)^\omega)$ are not postliminal even though $C^*(\mathbb{Z}^2) \cong C(\mathbb{T}^2)$ has continuous trace.

4. A reduced version of the decomposition theorem

The goal of this section is to prove a version of Theorem 3.2 for reduced crossed products. Let E be a second-countable locally compact Hausdorff groupoid with a Haar system β, and $\sigma : E^{(2)} \to \mathbb{T}$ be a continuous 2-cocycle. Let $u \in E^{(0)}$. The *left-regular representation* Π^u is the representation of $C_c(E,\sigma)$ on $L^2(E,\beta_u)$ characterized by

$$\langle \Pi^u(f)\xi,\zeta\rangle = \int_E\int_E f(\gamma\eta)\xi(\eta^{-1})\overline{\zeta(\gamma)}\sigma(\gamma\eta,\eta^{-1})\,d\beta^u(\gamma)\,d\beta_u(\eta) \tag{4.1}$$

for $f \in C_c(E,\sigma)$ and $\xi,\zeta \in L^2(E,\beta_u)$. Since Π^u is continuous in the inductive limit topology, it extends to a representation of $C^*(E,\sigma)$. The reduced C^*-algebra $C_r^*(E,\sigma)$ of (E,σ) is the completion of $C_c(E,\sigma)$ with respect to the norm $\|f\|_r = \sup_{u\in E^{(0)}}\{\|\Pi^u(f)\|\}$. Alternatively, $C_r^*(E,\sigma) = C^*(E,\sigma)/I$ where $I = \bigcap_{u\in E^{(0)}} \ker(\Pi^u)$; we write $q = q_{E}$ for the quotient map.

Theorem 4.1. *Let G be a second-countable, locally compact Hausdorff groupoid with a Haar system λ. Let $\omega : G^{(2)} \to \mathbb{T}$ be a continuous 2-cocycle and G^ω the extension associated to (G,ω). Let $\Upsilon : C^*(G^\omega) \to \oplus_{n\in\mathbb{Z}} C^*(G,\omega^n)$ and $\Upsilon_n : C^*(G^\omega) \to C^*(G,\omega^n)$ be as in Theorem 3.2. Then there exists a homomorphism $\Omega_n : C_r^*(G^\omega) \to C_r^*(G,\omega^n)$ such that the following diagram*

$$\begin{tikzcd} C^*(G^\omega) \arrow[r, "\Upsilon_n"] \arrow[d, "q_{G^\omega}"] & C^*(G,\omega^n) \arrow[d, "q_{G,n}"] \\ C_r^*(G^\omega) \arrow[r, "\Omega_n"] & C_r^*(G,\omega^n) \end{tikzcd} \tag{4.2}$$

commutes. Furthermore, the map $\Omega : C_r^*(G^\omega) \to \oplus_{n\in\mathbb{Z}} C_r^*(G,\omega^n)$, defined by $a \mapsto (\Omega_n(a))$, is an isomorphism.

For $n \in \mathbb{Z}$ and $u \in G^{(0)}$, we write L_n^u for the left-regular representation of $C^*(G,\omega^n)$ on $L^2(G,\lambda_u)$ and R^u for the left-regular representation of $C^*(G^\omega)$ on $L^2(G^\omega,\tau\times\lambda_u)$; both are characterized by (4.1).

Lemma 4.2. *Let G^ω be the extension associated to (G,ω). Let $u \in G^{(0)}$. For $n \in \mathbb{Z}$ define*

$$\mathcal{H}_n^u := \overline{\operatorname{span}}\{s^{-n} \otimes \xi : \xi \in C_c(G)\} \subset L^2(G^\omega, \tau \times \lambda_u).$$

(a) *If $m \neq n$ then $\mathcal{H}_m^u$ is orthogonal to $\mathcal{H}_n^u$.*

(b) *There is a unitary $V_n : L^2(G,\lambda_u) \to \mathcal{H}_n^u$ such that*

$$V_n(\xi) = s^{-n} \otimes \xi \quad \textit{for} \quad \xi \in C_c(G). \tag{4.3}$$

(c) *There is a unitary $V : \oplus_{n\in\mathbb{Z}} L^2(G,\lambda_u) \to L^2(G^\omega, \tau \times \lambda_u)$ characterized by*

$$V((\xi_n)) = \oplus_{n\in\mathbb{Z}} V_n(\xi_n) \quad \textit{for} \quad \xi_n \in C_c(G).$$

(d) *For $n \in \mathbb{Z}$ let $L_n^u : C^*(G,\omega^n) \to B(L^2(G,\lambda_u))$ and $R^u : C^*(G^\omega) \to B(L^2(G^\omega, \tau \times \lambda_u))$ be the respective left-regular representations, and set $L^u = \oplus_{n\in\mathbb{Z}} L_n^u$. Then*

$$V(L^u \circ \Upsilon(a))V^* = R^u(a) \quad \textit{for all} \quad a \in C^*(G^\omega).$$

Proof. We compute:

$$\begin{aligned}
\left\langle r^{-m} \otimes \xi, r^{-n} \otimes \zeta \right\rangle_{L^2(G^\omega)} &= \int_G \int_{\mathbb{T}} r^{-m} \otimes \xi(t,\gamma) \overline{r^{-n} \otimes \zeta(t,\gamma)}\, dt\, d\lambda_u(\gamma) \\
&= \int_G \int_{\mathbb{T}} t^{n-m} \xi(\gamma) \overline{\zeta(\gamma)}\, dt\, d\lambda_u(\gamma) \\
&= \langle \xi, \zeta \rangle_{L^2(G)} \delta_{m,n}.
\end{aligned} \tag{4.4}$$

Now (4.4) implies, first, that $\mathcal{H}_m^u$ is orthogonal to $\mathcal{H}_n^u$, and second, that there is an isometry V_n satisfying (4.3). By definition of $\mathcal{H}_n^u$, V_n is onto and hence is unitary. This gives (a) and (b).

By Lemma 3.6, $\operatorname{span}\{r^m \otimes \xi : m \in \mathbb{Z}, \xi \in C_c(G)\}$ is dense in $C_c(G^\omega)$ in the inductive limit topology, and hence it is dense in $L^2(G^\omega, \tau \times \lambda_u)$ as well. Now (c) follows from (a) and (b).

For (d), let $m,n \in \mathbb{Z}$, $\xi, \zeta \in C_c(G)$ and $F \in C_c(G^\omega)$. Then, using Fubini's Theorem several times,

$$\begin{aligned}
&\left\langle R^u(F)(r^{-m} \otimes \xi), r^{-n} \otimes \zeta \right\rangle_{L^2(G^\omega)} \\
&= \int_G \int_{\mathbb{T}} \int_G \int_{\mathbb{T}} F\big((t,\gamma)(s,\eta)\big)(r^{-m} \otimes \xi)\big((s,\eta)^{-1}\big) \overline{r^{-n} \otimes \zeta(t,\gamma)}\, ds\, d\lambda^u(\eta)\, dt\, d\lambda_u(\gamma) \\
&= \int_G \int_G \int_{\mathbb{T}} \int_{\mathbb{T}} F(st\omega(\gamma,\eta), \gamma\eta) s^m \omega(\eta,\eta^{-1})^m \xi(\eta^{-1}) t^n \overline{\zeta(\gamma)}\, ds\, dt\, d\lambda^u(\eta)\, d\lambda_u(\gamma)
\end{aligned}$$

and, replacing s with $st^{-1}\overline{\omega(\gamma,\eta)}$, gives

$$= \int_G \int_G \int_{\mathbb{T}} \int_{\mathbb{T}} F(s,\gamma\eta) s^m t^{-m} \overline{\omega(\gamma,\eta)}^m \omega(\eta,\eta^{-1})^m \xi(\eta^{-1}) t^n \overline{\zeta(\gamma)}\, ds\, dt\, d\lambda^u(\eta)\, d\lambda_u(\gamma)$$

which, because $\omega(\gamma,\eta)\omega(\gamma\eta,\eta^{-1}) = \omega(\eta,\eta^{-1})\omega(\gamma,\eta\eta^{-1}) = \omega(\eta,\eta^{-1})$, becomes

$$\begin{aligned}
&= \int_G \int_G \left(\int_{\mathbb{T}} F(s,\gamma\eta) s^m\, ds \right) \left(\int_{\mathbb{T}} t^{n-m}\, dt \right) \omega(\gamma\eta,\eta^{-1})^m \xi(\eta^{-1}) \overline{\zeta(\gamma)}\, d\lambda^u(\eta)\, d\lambda_u(\gamma) \\
&= \delta_{m,n} \int_G \int_G \Upsilon_m(F)(\gamma\eta) \xi(\eta^{-1}) \overline{\zeta(\gamma)} \omega(\gamma\eta,\eta^{-1})^m\, d\lambda^u(\eta)\, d\lambda_u(\gamma) \\
&= \delta_{m,n} \left\langle L_m^u(\Upsilon_m(F))\xi, \zeta \right\rangle_{L^2(G)}.
\end{aligned}$$

Since $C_c(G^\omega)$ is dense in $C^*(G^\omega)$, it follows that for $a \in C^*(G^\omega)$

$$\begin{aligned}\left\langle R^u(a)\Big(\sum r^{-m}\otimes\xi_m\Big), \sum r^{-n}\otimes\zeta_n\right\rangle_{L^2(G^\omega)} &= \sum_{m,n}\left\langle L^u_m(\Upsilon_m(a))\xi_m,\zeta_n\right\rangle_{L^2(G)}\delta_{m,n}\\ &= \sum_n \left\langle L^u_n(\Upsilon_n(a))\xi_n,\zeta_n\right\rangle_{L^2(G)}.\end{aligned}$$

So for $x=\sum s^m\otimes\xi_m$, $y=\sum s^n\otimes\zeta_n$ we have

$$\left\langle R^u(a)x,y\right\rangle_{L^2(G^\omega)} = \sum_n \left\langle L^u_n(\Upsilon_n(a))\xi_n,\zeta_n\right\rangle_{L^2(G)} = \left\langle L^u(\Upsilon(a))V^*x,V^*y\right\rangle_{\oplus_{n\in\mathbb{Z}}L^2(G)},$$

and then (d) follows because the set of such x, y is dense in $L^2(G^\omega, \tau\times\lambda_u)$. □

Proof of Theorem 4.1. By Lemma 4.2(d), we have $\ker(R^u)\subset\ker(L^u_n\circ\Upsilon_n)$ for all n. Since this holds for all $u\in G^{(0)}$, $\ker(q_{G^\omega})\subset\ker(q_{G,n})$. Thus the map $q_{G,n}\circ\Upsilon_n$ induces a homomorphism Ω_n such that the diagram (4.2) commutes.

To see that $\Omega=(\Omega_n)$ is isometric, recall from Proposition 3.7 that $C^*(G^\omega) = \oplus_{m\in\mathbb{Z}}I_m$ and let $a=(a_n)\in C^*(G^\omega)$ where $a_n\in I_n$. Using first Lemma 4.2(d), and second, $\Upsilon_n=\Upsilon|_{I_n}$ and $L^u=\oplus_n L^u_n$, we get

$$\|R^u(a)\| = \|L^u(\Upsilon(a))\| = \max_n \|L^u_n(\Upsilon_n(a_n))\|.$$

Since this holds for all $u\in G^{(0)}$,

$$\begin{aligned}\|q_{G^\omega}(a)\|_{C^*_r(G^\omega)} &= \max_n \|q_{G,n}(\Upsilon_n(a_n))\|_{C^*_r(G,\omega^n)} = \max_n \|\Omega_n(q_{G^\omega}(a_n))\|_{C^*_r(G,\omega^n)}\\ &= \|\Omega(q_{G^\omega}(a))\|_{C^*_r(G,\omega^n)}.\end{aligned}$$

Hence Ω is isometric. That Ω is surjective follows from the commutativity of the diagram since $\Upsilon=(\Upsilon_n)$ and the quotient maps are surjective. Thus Ω is an isomorphism. □

Corollary 4.3. *Let G^ω be the extension associated to (G,ω). If G is amenable, then $C^*(G,\omega)=C^*_r(G,\omega)$.*

Proof. Let $j: G^\omega\to G$ be the quotient map. Then $\ker j=\mathbb{T}\times G^{(0)}$ is amenable. Since G is amenable, Proposition 5.1.2 of [1] implies that G^ω is amenable. By Theorems 3.2 and 4.1 we have

$$\oplus_{n\in\mathbb{Z}}C^*_r(G,\omega^n)\cong C^*_r(G^\omega)=C^*(G^\omega)\cong\oplus_{n\in\mathbb{Z}}C^*(G,\omega^n).$$

By the commutativity of (4.2) the summands corresponding to $n=1$ match up, so the result follows. □

5. Actions of proper groupoids and fixed-point algebras

Let G be a principal proper groupoid. Then $G^\omega=\mathbb{T}\times_\omega G$ is also proper. There is an action lt of G^ω on $C_0(G^{(0)})$ defined by

$$\mathrm{lt}_\gamma(f)(v)=f(s_{G^\omega}(\gamma))\quad\text{ for } f\in C_0(G^{(0)}) \text{ and } \gamma\in G^\omega \text{ with } r_{G^\omega}(\gamma)=v.$$

Since $G^\omega\backslash G^{(0)}=G\backslash G^{(0)}$, [12, Proposition 2.2] implies that $C_0(G^\omega\backslash G^{(0)})$ is Morita equivalent to $C^*(G)$. Theorem 3.9 of [2] implies that $C_0(G^\omega\backslash G^{(0)})$ is Morita equivalent to an ideal I of $C^*(G^\omega)$. In the following proposition we reconcile these two results by using the decomposition of $C^*(G^\omega)$ into the direct sum $\oplus_{n\in\mathbb{Z}}C^*(G,\omega^n)$ to identify the ideal I with the summand corresponding to $n=0$.

Proposition 5.1. *Let G be a principal and proper groupoid. Then the generalized fixed-point algebra $C_0(G^{(0)})^{\mathrm{lt}} = C_0(G^\omega \backslash G^{(0)})$ is Morita equivalent to the direct summand $C^*(G) = C^*(G, \omega^0)$ of $C^*(G^\omega)$.*

Proof. Theorem 3.9 of [2] says that there is a C^*-subalgebra I of the reduced groupoid crossed product $C_0(G^{(0)}) \rtimes_{\mathrm{lt},r} G^\omega$ that is Morita equivalent to a generalized fixed-point algebra $C_0(G^{(0)})^{\mathrm{lt}}$ of $(C_0(G^{(0)}), G^\omega, \mathrm{lt})$. In our special case where the groupoid acts properly on its unit space, I is an ideal by Remark 4.14 of [2]. By Proposition 4.1 of [2], $C_0(G^{(0)})^{\mathrm{lt}} = C_0(G^\omega \backslash G^{(0)})$. Combining [1, Corollary 2.1.7 and Proposition 3.3.5] gives that G^ω is measurewise amenable, and hence $C_0(G^{(0)}) \rtimes_{\mathrm{lt},r} G^\omega = C_0(G^{(0)}) \rtimes_{\mathrm{lt}} G^\omega$ by [1, Proposition 6.1.10]. By [7, Remark 4.22], $C_0(G^{(0)}) \rtimes_{\mathrm{lt}} G^\omega$ is isomorphic to $C^*(G^\omega)$. So I is an ideal in $C^*(G^\omega)$ that is Morita equivalent to $C_0(G^\omega \backslash G^{(0)})$; it remains to identify the ideal I.

The imprimitivity bimodule implementing the Morita equivalence is a completion of $C_c(G^{(0)})$ with respect to the left inner product given by

$$ {}_I\langle f, g\rangle (t,\gamma) = f(r_{G^\omega}(t,\gamma))\overline{g(s_{G^\omega}(t,\gamma))} = f(r_G(\gamma))\overline{g(s_G(\gamma))} \tag{5.1} $$

for $f, g \in C_c(G^{(0)})$ and $(t,\gamma) \in \mathbb{T} \times_\omega G$. The point is that the inner product is independent of t. Thus I is an ideal of $C^*(G^\omega, 0) \cong C^*(G)$. When we apply Theorem 3.9 of [2] to the action lt of G on $C_0(G^{(0)})$ we obtain a Morita equivalence based on $C_c(G^{(0)})$ between an ideal J of $C_r^*(G)$ and $C_0(G \backslash G^{(0)})$, with left inner product given by

$$ {}_J\langle f', g'\rangle (\gamma) = f'(r_G(\gamma))\overline{g'(s_G(\gamma))} \tag{5.2} $$

for $f', g' \in C_c(G^{(0)})$ and $\gamma \in G$. Note that $C^*(G) = C_r^*(G)$ by amenability. Comparing (5.1) and (5.2) shows $I = J$. Finally, by [2, Theorem 5.9], $J = C^*(G)$. □

The action lt is called *saturated* if the ideal I of $C_0(G^{(0)}) \rtimes_{\mathrm{lt},r} G^\omega$ is in fact $C_0(G^{(0)}) \rtimes_{\mathrm{lt},r} G^\omega$. We note that in the situation of Proposition 5.1 the action is very far away from being saturated since it is just one summand in $\oplus_{n\in\mathbb{Z}} C^*(G, \omega^n)$.

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